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# Relations between two operator inequalities and their applications to paranormal operators

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## 1 Introduction

This report is based on the following preprint:

T.Yamazaki and M.Yanagida, *Relations between two operator inequalities and their applications to paranormal operators*, preprint.

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space  $H$ . An operator  $T$  is said to be positive (denoted by  $T \geq 0$ ) if  $(Tx, x) \geq 0$  for all  $x \in H$ . The following Theorem F is well known as a recent development on order preserving operator inequalities.

**Theorem F (Furuta inequality [11]).**

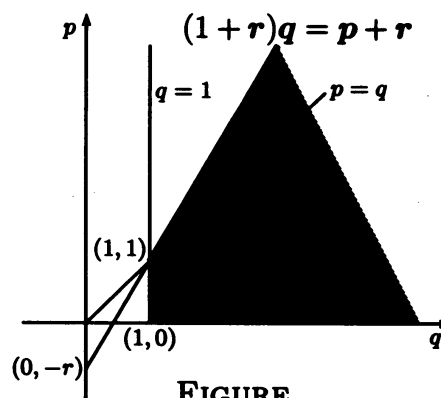
If  $A \geq B \geq 0$ , then for each  $r \geq 0$ ,

$$(i) \quad (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

$$(ii) \quad (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for  $p \geq 0$  and  $q \geq 1$  with  $(1+r)q \geq p+r$ .



FIGURE

Theorem F yields the famous Löwner-Heinz theorem “ $A \geq B \geq 0$  ensures  $A^\alpha \geq B^\alpha$  for any  $\alpha \in [0, 1]$ ” by putting  $r = 0$  in (i) or (ii) of Theorem F. Alternative proofs of Theorem F are given in [6] and [18], and also an elementary one page proof in [12]. It

was shown in [19] that the domain drawn for  $p, q$  and  $r$  in the Figure is the best possible for Theorem F.

For positive invertible operators  $A$  and  $B$ , the order defined by  $\log A \geq \log B$  is called the chaotic order. The chaotic order is weaker than the usual order since  $\log t$  is an operator monotone function. The following result is a characterization of the chaotic order which is an application of Theorem F.

**Theorem 1.A** ([7][13]). *For positive invertible operators  $A$  and  $B$ , the following assertions are mutually equivalent:*

- (i)  $\log A \geq \log B$ .
- (ii)  $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$  for all  $p \geq 0$  and  $r \geq 0$ .
- (iii)  $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$  for all  $p \geq 0$  and  $r \geq 0$ .

The case  $p = r$  of Theorem 1.A was shown in [4]. An alternative proof of Theorem 1.A was shown in [8], and also a breathtakingly simple proof in [21]. It was attempted in [22] to remove the invertibility of operators in Theorem 1.A.

Recently, Ito-Yamazaki [17] showed the following result on the relations between the two inequalities in Theorem 1.A.

**Theorem 1.B** ([17]). *Let  $A$  and  $B$  be positive operators. Then for each  $p > 0$  and  $r \geq 0$ , the following assertions hold:*

- (i) *If  $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ , then  $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$ .*
- (ii) *If  $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$  and  $N(A) \subseteq N(B)$ , then  $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ .*

It turns out by the following Lemma F that the two inequalities in Theorem 1.B are equivalent in case  $A$  and  $B$  are invertible.

**Lemma F** ([14]). *Let  $A$  be a positive invertible operator and  $B$  be an invertible operator. Then*

$$(BAB^*)^\lambda = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*$$

*holds for any real number  $\lambda$ .*

In fact, for each  $p \geq 0$  and  $r \geq 0$ ,

$$\begin{aligned} A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}} &\iff A^p \geq A^{\frac{p}{2}} B^{\frac{r}{2}} (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} B^{\frac{r}{2}} A^{\frac{p}{2}} && \text{by Lemma F} \\ &\iff B^{-r} \geq (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \\ &\iff (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r. \end{aligned}$$

## 2 Relations between two operator inequalities

As a parallel result to Theorem 1.B, we obtain the following result.

**Theorem 2.1.** *Let  $A$  and  $B$  be positive operators. Then for each  $p > 0$ ,  $r \geq 0$  and  $\lambda > 0$ , the following assertions hold:*

- (i) *If  $\frac{rB^{\frac{r}{2}}A^pB^{\frac{r}{2}} + p\lambda^{p+r}I}{(p+r)\lambda^p} \geq B^r$ , then  $A^p \geq \frac{(p+r)\lambda^p A^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{rA^{\frac{p}{2}}B^rA^{\frac{p}{2}} + p\lambda^{p+r}I}$ .*
- (ii) *If  $A^p \geq \frac{(p+r)\lambda^p A^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{rA^{\frac{p}{2}}B^rA^{\frac{p}{2}} + p\lambda^{p+r}I}$  and  $N(A) \subseteq N(B)$ , then  $\frac{rB^{\frac{r}{2}}A^pB^{\frac{r}{2}} + p\lambda^{p+r}I}{(p+r)\lambda^p} \geq B^r$ .*

We remark that the two inequalities in Theorem 2.1 are equivalent in case  $A$  and  $B$  are invertible. In fact, for each  $p \geq 0$ ,  $r \geq 0$  and  $\lambda > 0$ ,

$$\begin{aligned} A^p \geq \frac{(p+r)\lambda^p A^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{rA^{\frac{p}{2}}B^rA^{\frac{p}{2}} + p\lambda^{p+r}I} &\iff A^p \geq \frac{(p+r)\lambda^p}{rI + p\lambda^{p+r}A^{-\frac{p}{2}}B^{-r}A^{-\frac{p}{2}}} \\ &\iff \frac{rI + p\lambda^{p+r}A^{-\frac{p}{2}}B^{-r}A^{-\frac{p}{2}}}{(p+r)\lambda^p} \geq A^{-p} \\ &\iff \frac{rB^{\frac{r}{2}}A^pB^{\frac{r}{2}} + p\lambda^{p+r}I}{(p+r)\lambda^p} \geq B^r. \end{aligned}$$

We also remark that the inequalities in Theorem 2.1 are weaker than those in Theorem 1.B. In fact, by the arithmetic-geometric-harmonic mean inequality,

$$\begin{aligned} (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} &= \left( \frac{B^{\frac{r}{2}}A^pB^{\frac{r}{2}}}{\lambda^p} \right)^{\frac{r}{p+r}} (\lambda^r)^{\frac{p}{p+r}} \\ &\leq \frac{r}{p+r} \frac{B^{\frac{r}{2}}A^pB^{\frac{r}{2}}}{\lambda^p} + \frac{p}{p+r} \lambda^r I = \frac{rB^{\frac{r}{2}}A^pB^{\frac{r}{2}} + p\lambda^{p+r}I}{(p+r)\lambda^p} \end{aligned}$$

and

$$\begin{aligned} (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}} &= \left( \frac{A^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{\lambda^r} \right)^{\frac{p}{p+r}} (\lambda^p)^{\frac{r}{p+r}} \\ &\geq \left\{ \frac{p}{p+r} \left( \frac{A^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{\lambda^r} \right)^{-1} + \frac{r}{p+r} (\lambda^p I)^{-1} \right\}^{-1} = \frac{(p+r)\lambda^p A^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{rA^{\frac{p}{2}}B^rA^{\frac{p}{2}} + p\lambda^{p+r}I} \end{aligned}$$

hold for each positive invertible operators  $A$  and  $B$ ,  $p \geq 0$ ,  $r \geq 0$  and  $\lambda > 0$ . Hence Theorem 2.1 can be understood as a parallel result to Theorem 1.B.

In order to give a proof of Theorem 2.1, we use the following lemma.

**Lemma 2.A** ([17]). *Let  $A$  be a positive operator. Then*

$$\lim_{\varepsilon \rightarrow +0} A^{\frac{1}{2}}(A + \varepsilon I)^{-1}A^{\frac{1}{2}} = \lim_{\varepsilon \rightarrow +0} (A + \varepsilon I)^{-1}A = P_{N(A)^\perp}$$

holds, where  $P_{\mathcal{M}}$  is the projection onto a closed subspace  $\mathcal{M}$ .

*Proof of Theorem 2.1.*

*Proof of (i).* By the assumption,

$$A^{\frac{p}{2}} B^{\frac{r}{2}} (B^r + \varepsilon I)^{-1} B^{\frac{r}{2}} A^{\frac{p}{2}} \geq A^{\frac{p}{2}} B^{\frac{r}{2}} \left( \frac{r B^{\frac{r}{2}} A^p B^{\frac{r}{2}} + p \lambda^{p+r} I}{(p+r) \lambda^p} + \varepsilon I \right)^{-1} B^{\frac{r}{2}} A^{\frac{p}{2}}$$

holds for any  $\varepsilon > 0$ . By tending  $\varepsilon \rightarrow +0$  and Lemma 2.A, we have

$$A^p \geq A^{\frac{p}{2}} P_{N(B)^\perp} A^{\frac{p}{2}} \geq A^{\frac{p}{2}} B^{\frac{r}{2}} \left( \frac{r B^{\frac{r}{2}} A^p B^{\frac{r}{2}} + p \lambda^{p+r} I}{(p+r) \lambda^p} \right)^{-1} B^{\frac{r}{2}} A^{\frac{p}{2}} = \frac{(p+r) \lambda^p A^{\frac{p}{2}} B^r A^{\frac{p}{2}}}{r A^{\frac{p}{2}} B^r A^{\frac{p}{2}} + p \lambda^{p+r} I}$$

since

$$\begin{aligned} A^{\frac{p}{2}} B^{\frac{r}{2}} (r B^{\frac{r}{2}} A^p B^{\frac{r}{2}} + p \lambda^{p+r} I)^{-1} B^{\frac{r}{2}} A^{\frac{p}{2}} &= U |A^{\frac{p}{2}} B^{\frac{r}{2}}| (r |A^{\frac{p}{2}} B^{\frac{r}{2}}|^2 + p \lambda^{p+r} I)^{-1} |A^{\frac{p}{2}} B^{\frac{r}{2}}| U^* \\ &= U |A^{\frac{p}{2}} B^{\frac{r}{2}}|^2 U^* (r |B^{\frac{r}{2}} A^{\frac{p}{2}}|^2 + p \lambda^{p+r} I)^{-1} \\ &= |B^{\frac{r}{2}} A^{\frac{p}{2}}|^2 (r |B^{\frac{r}{2}} A^{\frac{p}{2}}|^2 + p \lambda^{p+r} I)^{-1} \\ &= \frac{A^{\frac{p}{2}} B^r A^{\frac{p}{2}}}{r A^{\frac{p}{2}} B^r A^{\frac{p}{2}} + p \lambda^{p+r} I}, \end{aligned}$$

where  $A^{\frac{p}{2}} B^{\frac{r}{2}} = U |A^{\frac{p}{2}} B^{\frac{r}{2}}|$  is the polar decomposition of  $A^{\frac{p}{2}} B^{\frac{r}{2}}$ .

*Proof of (ii).* By the assumption,

$$B^{\frac{r}{2}} A^{\frac{p}{2}} \left( \frac{(p+r) \lambda^p A^{\frac{p}{2}} B^r A^{\frac{p}{2}}}{r A^{\frac{p}{2}} B^r A^{\frac{p}{2}} + p \lambda^{p+r} I} + \varepsilon I \right)^{-1} A^{\frac{p}{2}} B^{\frac{r}{2}} \geq B^{\frac{r}{2}} A^{\frac{p}{2}} (A^p + \varepsilon I)^{-1} A^{\frac{p}{2}} B^{\frac{r}{2}}$$

holds for any  $\varepsilon > 0$ . By tending  $\varepsilon \rightarrow +0$  and Lemma 2.A, we have

$$\frac{r B^{\frac{r}{2}} A^p B^{\frac{r}{2}} + p \lambda^{p+r} I}{(p+r) \lambda^p} \geq \frac{r B^{\frac{r}{2}} A^p B^{\frac{r}{2}} + p \lambda^{p+r} P_{N(A^{\frac{p}{2}} B^{\frac{r}{2}})^\perp}}{(p+r) \lambda^p} \geq B^{\frac{r}{2}} P_{N(A)^\perp} B^{\frac{r}{2}} \geq B^r$$

since  $N(A) \subseteq N(B)$  is equivalent to  $P_{N(A)^\perp} \geq P_{N(B)^\perp}$  and

$$\begin{aligned} \lim_{\varepsilon \rightarrow +0} B^{\frac{r}{2}} A^{\frac{p}{2}} \left( \frac{A^{\frac{p}{2}} B^r A^{\frac{p}{2}}}{r A^{\frac{p}{2}} B^r A^{\frac{p}{2}} + p \lambda^{p+r} I} + \frac{\varepsilon I}{(p+r) \lambda^p} \right)^{-1} A^{\frac{p}{2}} B^{\frac{r}{2}} \\ &= \lim_{\varepsilon \rightarrow +0} a(\varepsilon) B^{\frac{r}{2}} A^{\frac{p}{2}} \left( \frac{A^{\frac{p}{2}} B^r A^{\frac{p}{2}} + b(\varepsilon) I}{r A^{\frac{p}{2}} B^r A^{\frac{p}{2}} + p \lambda^{p+r} I} \right)^{-1} A^{\frac{p}{2}} B^{\frac{r}{2}} \\ &= \lim_{\varepsilon \rightarrow +0} a(\varepsilon) V \frac{|B^{\frac{r}{2}} A^{\frac{p}{2}}| (|B^{\frac{r}{2}} A^{\frac{p}{2}}|^2 + b(\varepsilon) I)^{-1} |B^{\frac{r}{2}} A^{\frac{p}{2}}|}{(r |B^{\frac{r}{2}} A^{\frac{p}{2}}|^2 + p \lambda^{p+r} I)^{-1}} V^* \\ &= V \frac{P_{N(B^{\frac{r}{2}} A^{\frac{p}{2}})^\perp}}{(r |B^{\frac{r}{2}} A^{\frac{p}{2}}|^2 + p \lambda^{p+r} I)^{-1}} V^* \quad \text{by } a(0) = 1, \lim_{\varepsilon \rightarrow +0} b(\varepsilon) = 0 \text{ and Lemma 2.A} \\ &= V (r |B^{\frac{r}{2}} A^{\frac{p}{2}}|^2 + p \lambda^{p+r} P_{N(B^{\frac{r}{2}} A^{\frac{p}{2}})^\perp}) V^* \\ &= r |A^{\frac{p}{2}} B^{\frac{r}{2}}|^2 + p \lambda^{p+r} P_{N(A^{\frac{p}{2}} B^{\frac{r}{2}})^\perp} \\ &= r B^{\frac{r}{2}} A^p B^{\frac{r}{2}} + p \lambda^{p+r} P_{N(A^{\frac{p}{2}} B^{\frac{r}{2}})^\perp}, \end{aligned}$$

where  $B^{\frac{r}{2}} A^{\frac{p}{2}} = V |B^{\frac{r}{2}} A^{\frac{p}{2}}|$  is the polar decomposition of  $B^{\frac{r}{2}} A^{\frac{p}{2}}$ ,  $a(\varepsilon) = \frac{(p+r) \lambda^p}{(p+r) \lambda^p + \varepsilon r}$  and  $b(\varepsilon) = \frac{\varepsilon p \lambda^{p+r}}{(p+r) \lambda^p + \varepsilon r}$ . Therefore the proof is complete.

### 3 Classes of non-normal operators

In the following sections, we shall show applications of Theorem 2.1 to non-normal operators. To begin with, we introduce several classes of non-normal operators.

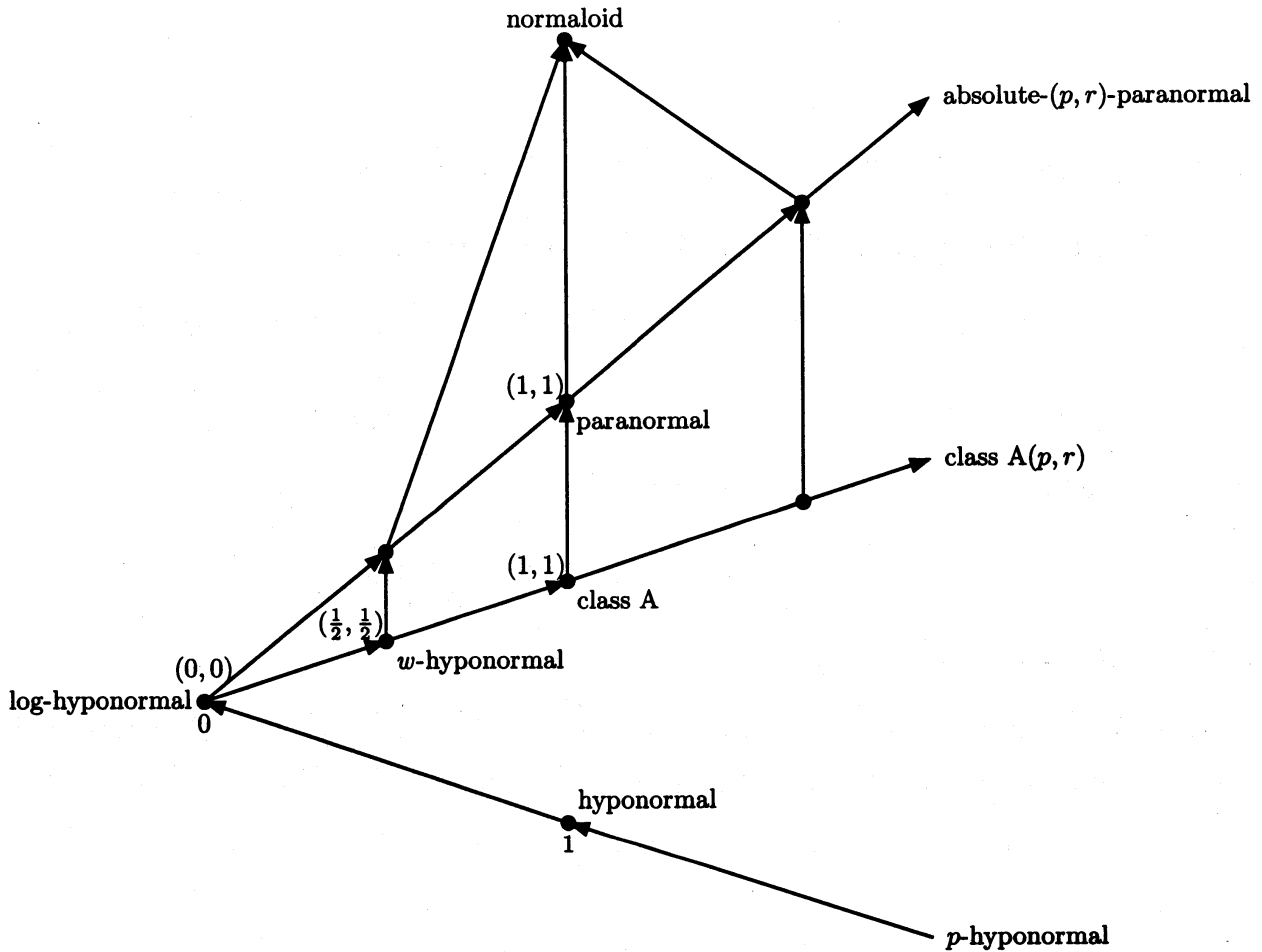
**Definition** ([2][9][10][15][16][23]). Let  $p > 0$  and  $r > 0$ .

- (i)  $T$  is  $p$ -hyponormal  $\iff (T^*T)^p \geq (TT^*)^p$ .
- (ii)  $T$  is log-hyponormal  $\iff T$  is invertible and  $\log T^*T \geq \log TT^*$ .
- (iii)  $T$  is hyponormal  $\iff T^*T \geq TT^* \iff T$  is 1-hyponormal.
- (iv)  $T$  belongs to class  $A(p, r)$   $\iff (|T^*|^r |T|^{2p} |T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r}$ .
- (v)  $T$  belongs to class  $A$   $\iff |T^2| \geq |T|^2 \iff T$  belongs to class  $A(1, 1)$ .
- (vi)  $T$  is  $w$ -hyponormal  $\iff |\tilde{T}| \geq |T| \geq |(\tilde{T})^*| \iff T$  belongs to class  $A(\frac{1}{2}, \frac{1}{2})$  ([17]).
- (vii)  $T$  is absolute- $(p, r)$ -paranormal  $\iff ||T|^p |T^*|^r x||^r \geq ||T^*|^r x||^{p+r}$  for all  $\|x\| = 1$ .
- (viii)  $T$  is paranormal  $\iff \|T^2 x\| \geq \|Tx\|^2$  for all  $\|x\| = 1$   
 $\iff T$  is absolute- $(1, 1)$ -paranormal.

Inclusion relations among these classes are as follows and can be expressed as the diagram on the next page.

**Theorem 3.A** ([9][17][23]).

- (i)  $T$  is  $p$ -hyponormal for some  $p > 0$  or log-hyponormal  
 $\implies T$  belongs to class  $A(p, r)$  for all  $p > 0$  and  $r > 0$ .
- (ii) For each  $p > 0$  and  $r > 0$ ,  
 $T$  belongs to class  $A(p, r) \implies T$  is absolute- $(p, r)$ -paranormal.
- (iii)  $T$  is absolute- $(p, r)$ -paranormal for some  $p > 0$  and  $r > 0$   
 $\implies T$  is normaloid (i.e.,  $\|T\| = r(T)$ ).
- (iv)  $T$  is log-hyponormal  
 $\iff T$  is invertible and absolute- $(p, p)$ -paranormal for all  $p > 0$   
 $\iff T$  is invertible and absolute- $(p, r)$ -paranormal for all  $p > 0$  and  $r > 0$ .
- (v) For each  $0 < p_1 \leq p_2$  and  $0 < r_1 \leq r_2$ ,  
 $T$  belongs to class  $A(p_1, r_1) \implies T$  belongs to class  $A(p_2, r_2)$ .
- (vi) For each  $0 < p_1 \leq p_2$  and  $0 < r_1 \leq r_2$ ,  
 $T$  is absolute- $(p_1, r_1)$ -paranormal  $\implies T$  is absolute- $(p_2, r_2)$ -paranormal.



## 4 Normality conditions via paranormality

Recently, Ito-Yamazaki [17] showed the following result on the normality of class  $A(p, r)$  operators.

**Theorem 4.A ([17]).** *Let  $p_1 > 0$ ,  $p_2 > 0$ ,  $r_1 > 0$  and  $r_2 > 0$ . If  $T$  belongs to class  $A(p_1, r_1)$  and  $T^*$  belongs to class  $A(p_2, r_2)$ , then  $T$  is normal.*

On the other hand, Ando [3] showed the following result on the normality of paranormal operators under the condition  $N(T) = N(T^*)$ .

**Theorem 4.B ([3]).** *If  $T$  and  $T^*$  are paranormal with  $N(T) = N(T^*)$ , then  $T$  is normal.*

We obtain the following result as an application of Theorem 2.1.

**Theorem 4.1.** *Let  $p_1 > 0$ ,  $p_2 > 0$ ,  $r_1 > 0$  and  $r_2 > 0$ . If  $T$  is absolute- $(p_1, r_1)$ -paranormal and  $T^*$  is absolute- $(p_2, r_2)$ -paranormal, then  $T$  is normal.*

Theorem 4.1 is an extension of Theorem 4.A by (ii) of Theorem 3.A. Theorem 4.1 is also an extension of Theorem 4.B since the following result can be obtained as a simple

corollary of Theorem 4.1 by putting  $p_1 = p_2 = r_1 = r_2 = 1$ . We remark that Corollary 4.2 requires no kernel conditions.

**Corollary 4.2.** *If  $T$  and  $T^*$  are paranormal, then  $T$  is normal.*

In order to give a proof of Theorem 4.1, we prepare the following results.

**Theorem 4.C** ([23]). *Let  $p > 0$  and  $r > 0$ .  $T$  is absolute- $(p, r)$ -paranormal if and only if*

$$r|T^*|^r|T|^{2p}|T^*|^r - (p+r)\lambda^p|T^*|^{2r} + p\lambda^{p+r}I \geq 0 \quad \text{for all } \lambda > 0.$$

**Theorem 4.D** ([3]). *Let  $A$  and  $B$  be positive operators. If*

$$\frac{A^2 + \lambda^2 I}{2\lambda} \geq B \quad \text{and} \quad B \geq \frac{2\lambda A^2}{A^2 + \lambda^2 I}$$

*hold for all  $\lambda > 0$ , then  $A = B$ .*

*Proof of Theorem 4.1.* Put  $k = \max\{p_1, p_2, r_1, r_2\}$ . If  $T$  is absolute- $(p_1, r_1)$ -paranormal, then  $T$  is absolute- $(k, k)$ -paranormal by (vi) of Theorem 3.A. By Theorem 4.C, we have

$$k|T^*|^k|T|^{2k}|T^*|^k - 2k\lambda^k|T^*|^{2k} + k\lambda^{2k}I \geq 0 \quad \text{for all } \lambda > 0.$$

This is equivalent to

$$\frac{|T^*|^k|T|^{2k}|T^*|^k + \lambda^{2k}I}{2\lambda^k} \geq |T^*|^{2k},$$

so that by (i) of Theorem 2.1, we have

$$\frac{|T^*|^k|T|^{2k}|T^*|^k + \lambda^{2k}I}{2\lambda^k} \geq |T^*|^{2k} \quad \text{and} \quad |T|^{2k} \geq \frac{2\lambda^k|T^*|^k|T^*|^{2k}|T|^k}{|T|^k|T^*|^{2k}|T|^k + \lambda^{2k}I}. \quad (4.1)$$

On the other hand, if  $T^*$  is absolute- $(p_2, r_2)$ -paranormal, then  $T^*$  is absolute- $(k, k)$ -paranormal by (vi) of Theorem 3.A. By Theorem 4.C, we have

$$k|T|^k|T^*|^{2k}|T|^k - 2k\lambda^k|T|^{2k} + k\lambda^{2k}I \geq 0 \quad \text{for all } \lambda > 0.$$

This is equivalent to

$$\frac{|T|^k|T^*|^{2k}|T|^k + \lambda^{2k}I}{2\lambda^k} \geq |T|^{2k},$$

so that by (i) of Theorem 2.1, we have

$$\frac{|T|^k|T^*|^{2k}|T|^k + \lambda^{2k}I}{2\lambda^k} \geq |T|^{2k} \quad \text{and} \quad |T^*|^{2k} \geq \frac{2\lambda^k|T^*|^k|T|^{2k}|T^*|^k}{|T^*|^k|T|^{2k}|T^*|^k + \lambda^{2k}I}. \quad (4.2)$$

Hence  $(|T^*|^k|T|^{2k}|T^*|^k)^{\frac{1}{2}} = |T^*|^{2k}$  and  $(|T|^k|T^*|^{2k}|T|^k)^{\frac{1}{2}} = |T|^{2k}$  by (4.1), (4.2) and Theorem 4.D, that is,  $T$  and  $T^*$  belong to class  $A(k, k)$ . Therefore  $T$  is normal by Theorem



## 5 Normality conditions via Aluthge transformation

Let  $T$  be an operator whose polar decomposition is  $T = U|T|$ . Then  $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$  is called Aluthge transformation of  $T$ . Aluthge transformation was firstly introduced in [1] and has been studied by many researchers.

Chō-Huruya-Kim [5] showed the following result on the normality of  $w$ -hyponormal operators via Aluthge transformation.

**Theorem 5.A ([5]).** *If  $T$  is  $w$ -hyponormal and  $\tilde{T}$  is normal, then  $T$  is also normal.*

We remark that Theorem 5.A can be considered as an extension of the following result since every log-hyponormal operator is  $w$ -hyponormal by (i) of Theorem 3.A and  $T_t = U|T|^{2t}$  is log-hyponormal for any  $t > 0$  if  $T = U|T|$  is log-hyponormal.

**Theorem 5.B ([20]).** *If  $T = U|T|$  is log-hyponormal and  $\tilde{T}_t = |T|^t U|T|^t$  is normal for some  $t > 0$ , then  $T$  is also normal.*

As an application of Theorem 2.1, we obtain the following result which is an extension of Theorem 5.A since every  $w$ -hyponormal operator is absolute- $(\frac{1}{2}, \frac{1}{2})$ -paranormal by (ii) of Theorem 3.A.

**Theorem 5.1.** *If  $T$  is absolute- $(\frac{1}{2}, \frac{1}{2})$ -paranormal and  $(\tilde{T})^*$  is hyponormal, then  $T$  is normal.*

*Proof.* If  $T$  is absolute- $(\frac{1}{2}, \frac{1}{2})$ -paranormal, then

$$\frac{|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}} + \lambda I}{2\lambda^{\frac{1}{2}}} \geq |T^*| \quad (5.1)$$

holds for all  $\lambda > 0$  by Theorem 4.C. Applying (i) of Theorem 2.1 to (5.1), we have

$$|T| \geq \frac{2\lambda^{\frac{1}{2}}|T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}}}{|T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}} + \lambda I}. \quad (5.2)$$

Let  $T = U|T|$  be the polar decomposition of  $T$ . Then by (5.1) and (5.2),

$$\begin{aligned} \frac{|\tilde{T}|^2 + \lambda I}{2\lambda^{\frac{1}{2}}} &= \frac{U^*|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}}U + \lambda I}{2\lambda^{\frac{1}{2}}} \geq U^* \left( \frac{|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}} + \lambda I}{2\lambda^{\frac{1}{2}}} \right) U \\ &\geq U^*|T^*|U = |T| \geq \frac{2\lambda^{\frac{1}{2}}|T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}}}{|T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}} + \lambda I} = \frac{2\lambda^{\frac{1}{2}}|(\tilde{T})^*|^2}{|(\tilde{T})^*|^2 + \lambda I}. \end{aligned} \quad (5.3)$$

Since  $f(t) = \frac{t + \lambda}{2\lambda^{\frac{1}{2}}}$  and  $g(t) = \frac{2\lambda^{\frac{1}{2}}t}{t + \lambda}$  are operator monotone,

$$\frac{|(\tilde{T})^*|^2 + \lambda I}{2\lambda^{\frac{1}{2}}} \geq \frac{|\tilde{T}|^2 + \lambda I}{2\lambda^{\frac{1}{2}}} \geq |T| \quad \text{and} \quad |T| \geq \frac{2\lambda^{\frac{1}{2}}|(\tilde{T})^*|^2}{|(\tilde{T})^*|^2 + \lambda I} \geq \frac{2\lambda^{\frac{1}{2}}|\tilde{T}|^2}{|\tilde{T}|^2 + \lambda I} \quad (5.4)$$

hold by (5.3) and the hyponormality of  $(\tilde{T})^*$ . By (5.4) and Theorem 4.D, we have  $|\tilde{T}| = |T| = |(\tilde{T})^*|$ , that is,  $T$  is  $w$ -hyponormal and  $\tilde{T}$  is normal. Hence  $T$  is normal by Theorem 5.A.  $\square$

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